Warsaw University of Technology

Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

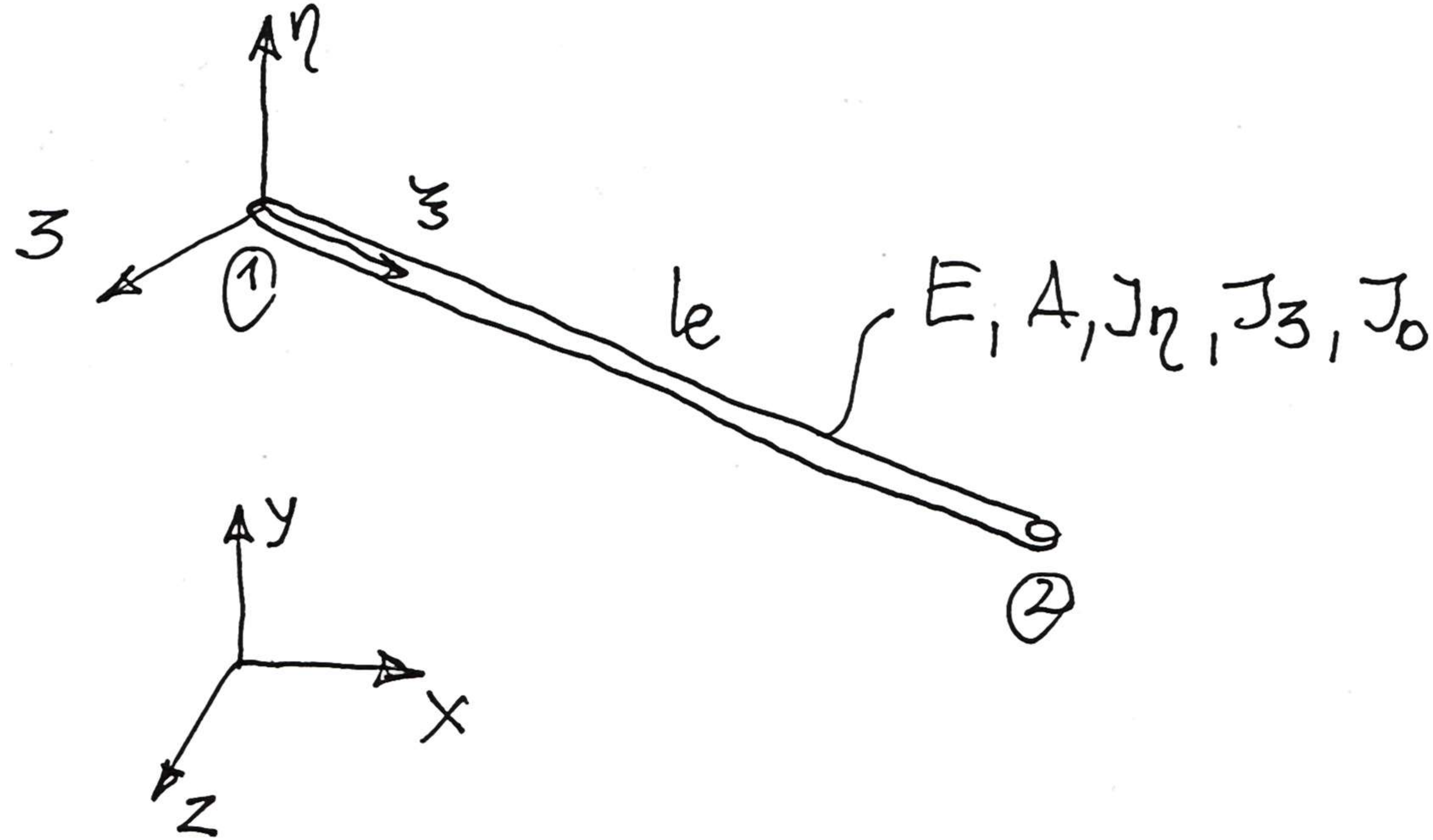
Institute of Aeronautics and Applied Mechanics

# Finite element method (FEM)

3D frame finite element

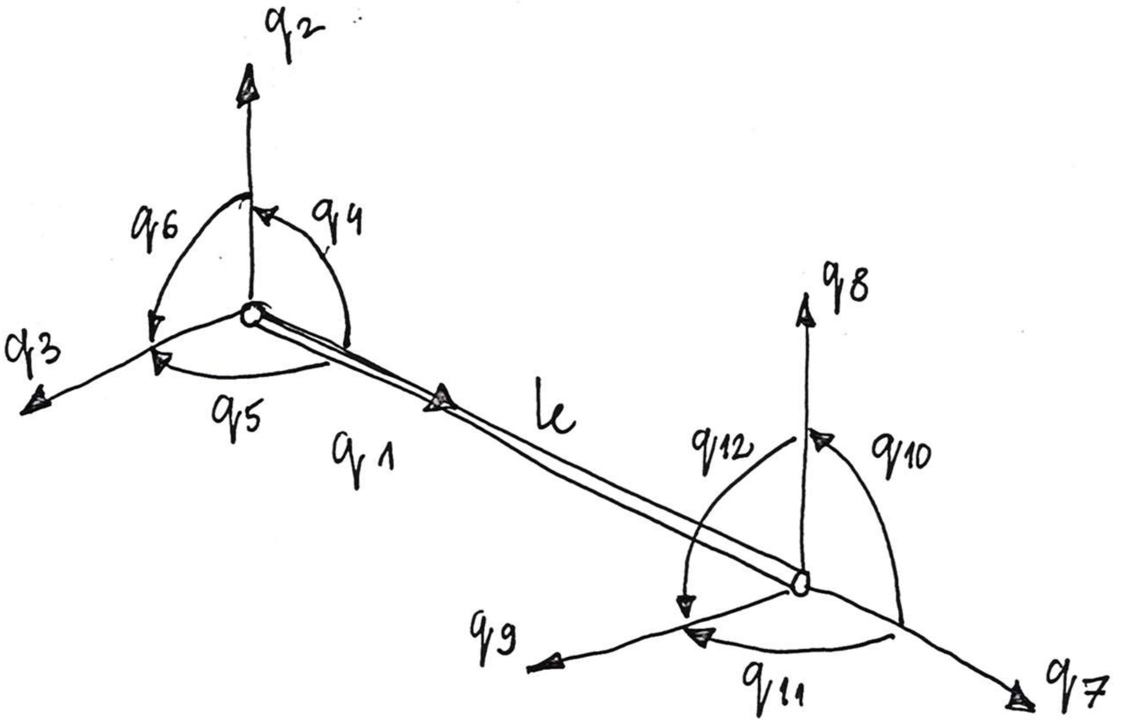
05.2021

# 3D FRAME ELEMENT



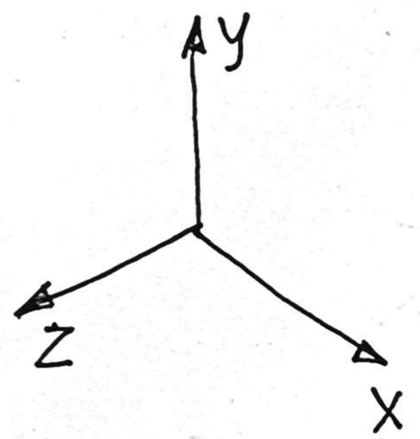
LOCAL PARAMETERS IN THE COORDINATE SYSTEM $\xi\eta\zeta$

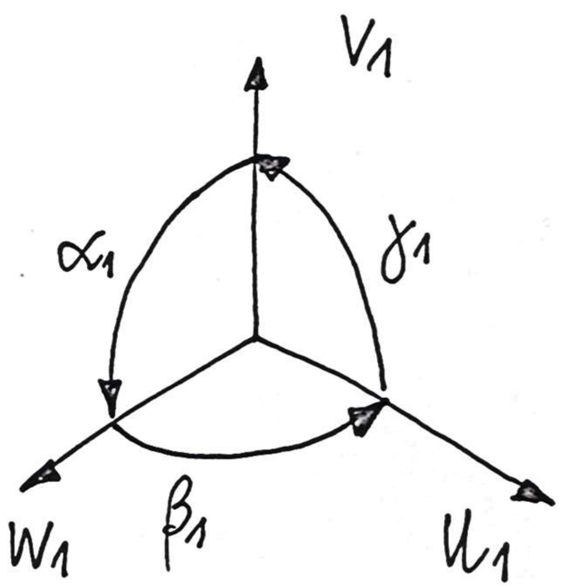
$$\underset{1\times 12}{\lfloor q \rfloor_e} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$



Local parameters in the coordinate system xyz

$$\lfloor q_g \rfloor_{e \atop 1\times 12} = \lfloor u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2 \rfloor$$



$$U_e = \frac{1}{2} \underset{1\times 12}{\lfloor q \rfloor_e} \, \underset{12\times 12}{[k]_e} \, \underset{12\times 1}{\{q\}_e} \qquad \text{where :}$$

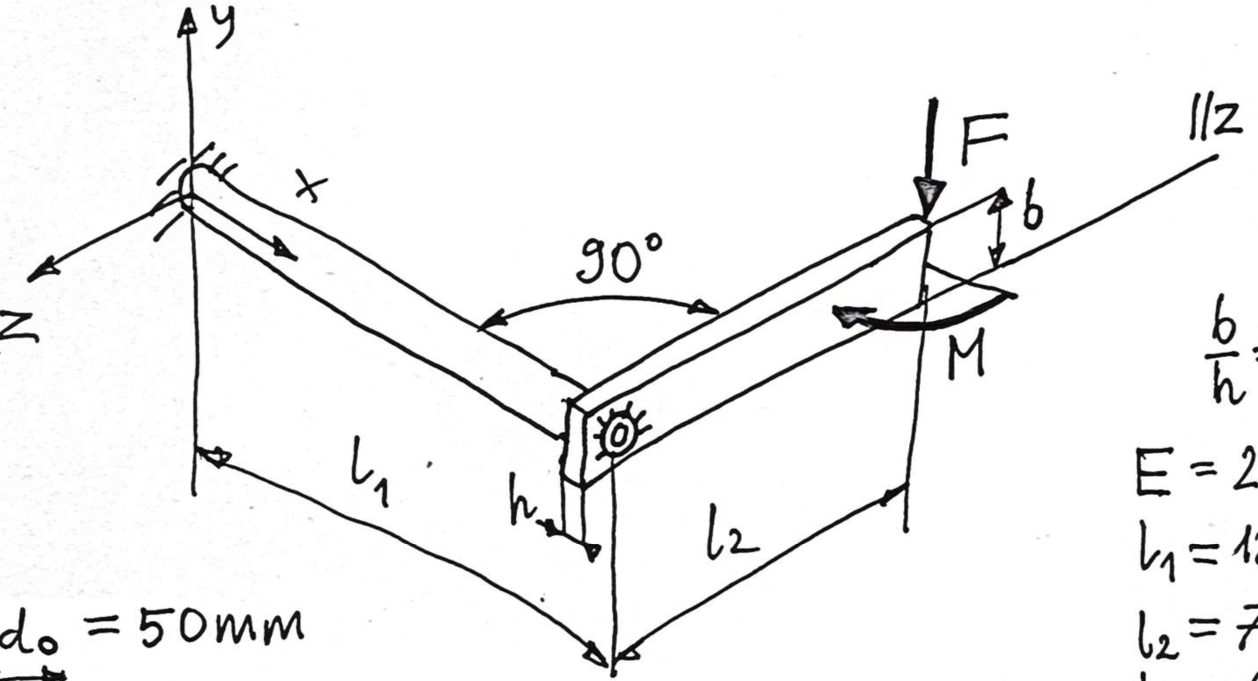
$$[k]_e = \begin{bmatrix}
a & & & & & & -a & & & & & \\
 & b & & d & & & & -b & & d & & \\
 & & c & & e & & & & -c & & e & \\
 & d & & 2r & & & & -d & & r & & \\
 & & e & & 2s & & & & -e & & s & \\
 & & & & & t & & & & & & -t \\
-a & & & & & & a & & & & & \\
 & -b & & -d & & & & b & & -d & & \\
 & & -c & & -e & & & & c & & -e & \\
 & d & & r & & & & -d & & 2r & & \\
 & & e & & s & & & & -e & & 2s & \\
 & & & & & -t & & & & & & t
\end{bmatrix}$$

$$a = \frac{EA}{L}, \quad b = \frac{12EJ_3}{l_e^3}, \quad c = \frac{12EJ_2}{l_e^3}, \quad d = \frac{6EJ_3}{l_e^2},$$

$$e = \frac{6EJ_2}{l_e^2}, \quad r = \frac{2EJ_3}{l_e}, \quad s = \frac{2EJ_2}{l_e}, \quad t = \frac{G \cdot J_0}{l_e}$$

$$\underset{}{U_e} = \underset{1\times 12}{\frac{1}{2} \lfloor q_g \rfloor_e} \cdot \underbrace{\underset{12\times 12}{[T_f]^T} \cdot \underset{12\times 12}{[k]_e} \cdot \underset{12\times 12}{[T_f]}}_{\underset{12\times 12}{[k_g]_e}} \cdot \underset{12\times 1}{\{q_g\}_e}$$

EXAMPLE : BUILD A FE MODEL USING 3D FRAME ELEMENTS. FIND UNKNOWN DISPLACEMENTS, STRESSES AND REACTIONS.




$d_0 = 50\,mm$

$t = 8\,mm$

$\frac{b}{h} = 2$

$E = 2 \cdot 10^5\,MPa$

$l_1 = 1200\,mm$

$l_2 = 750\,mm$

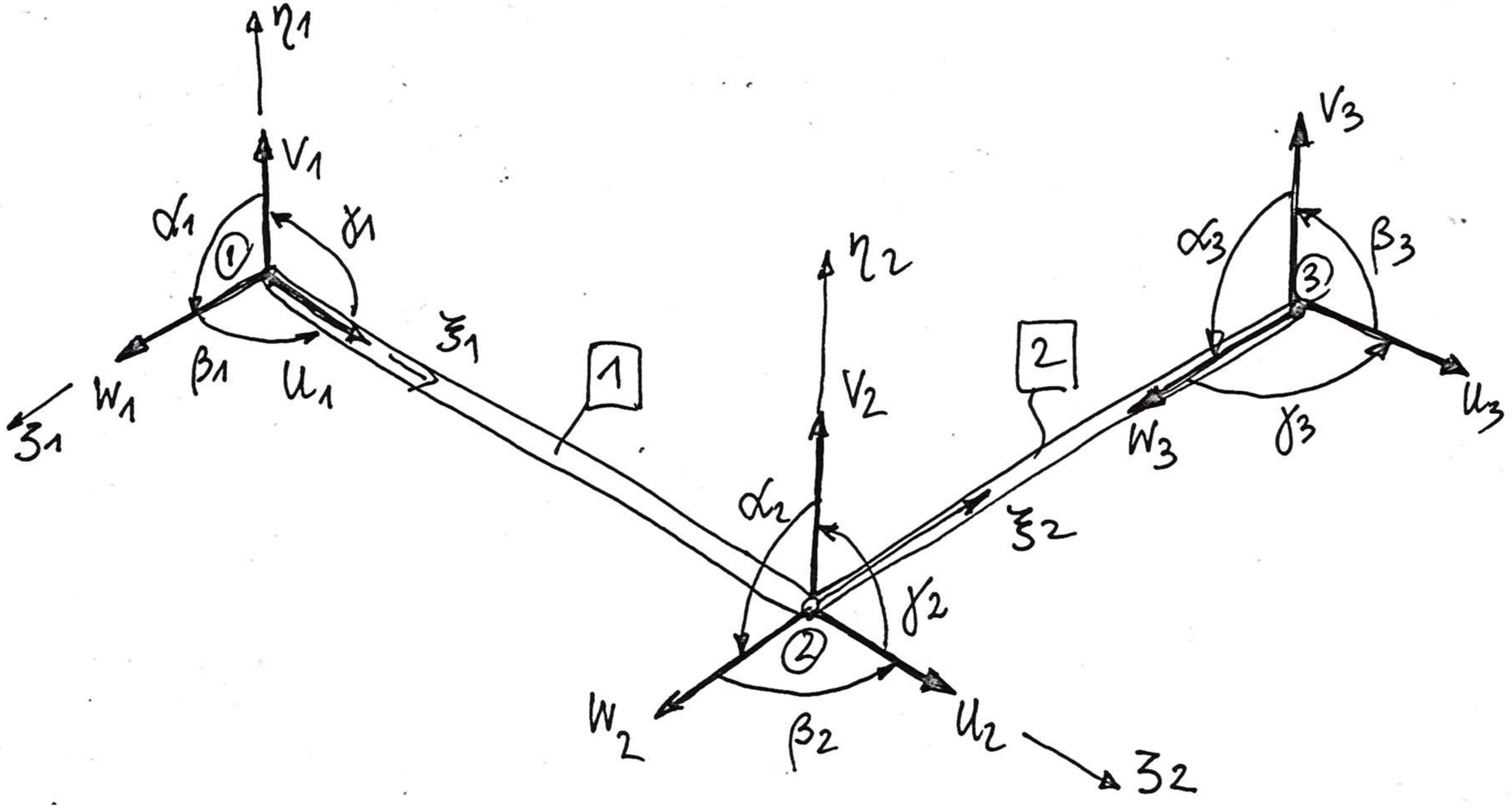
$b = 60\,mm$

$h = 30\,mm$

$F = 1000\,N$

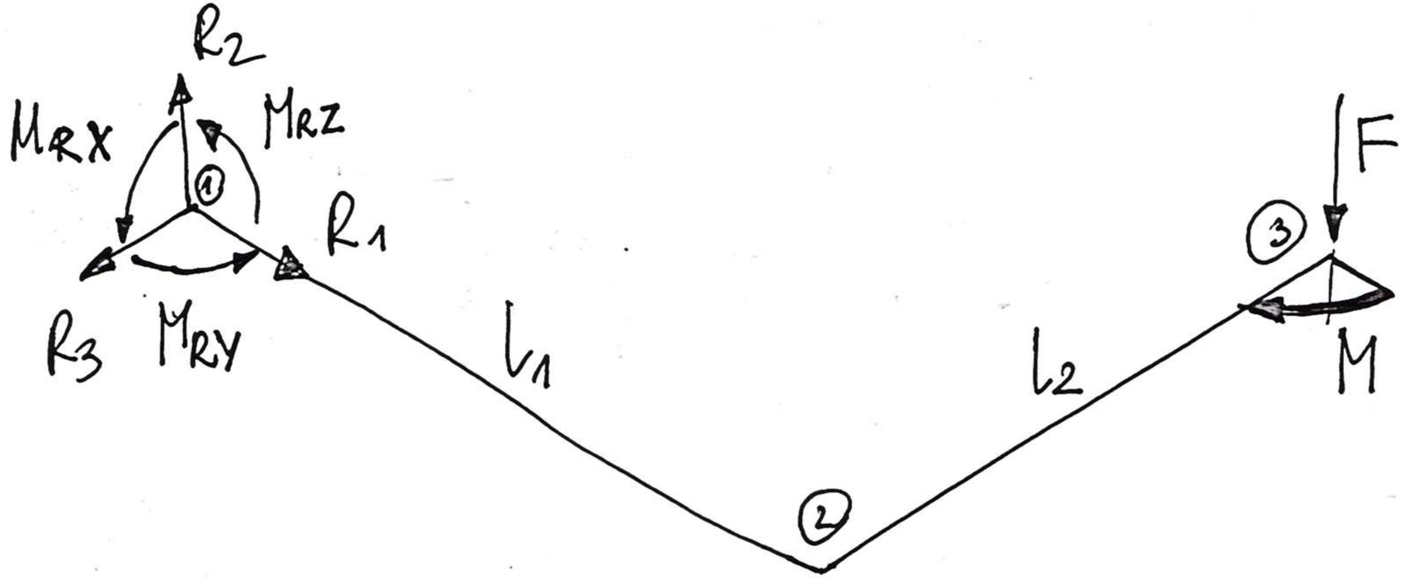
$M = 1000000\,Nmm$

# Global parameters in the coordinate system xyz




$$\underset{1\times 18}{\lfloor q \rfloor} = \lfloor u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2, u_3, v_3, w_3, \alpha_3, \beta_3, \gamma_3 \rfloor$$

# Global load vector




$$\lfloor F \rfloor_{1\times 18} = \lfloor R_1, R_2, R_3, M_{RX}, M_{RY}, M_{RZ}, 0, 0, 0, 0, 0, 0, 0, -F, 0, 0, -M, 0 \rfloor$$

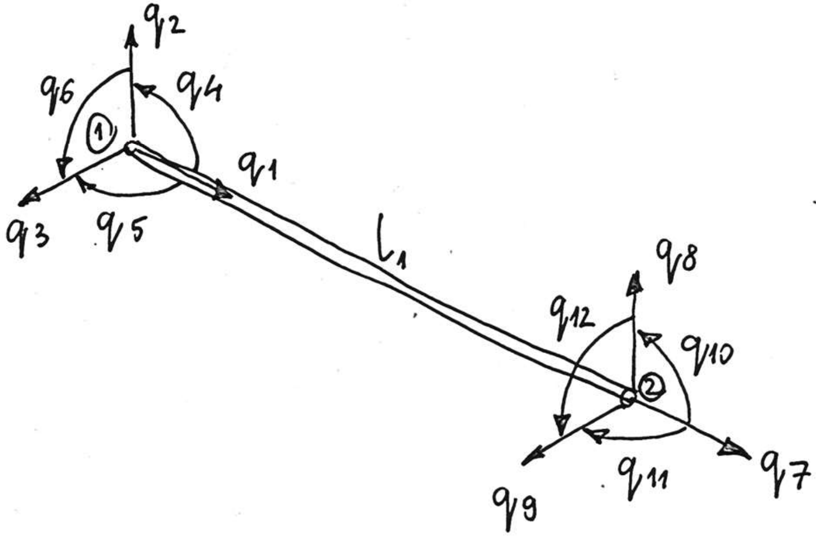
ELEMENT $\boxed{1}$:

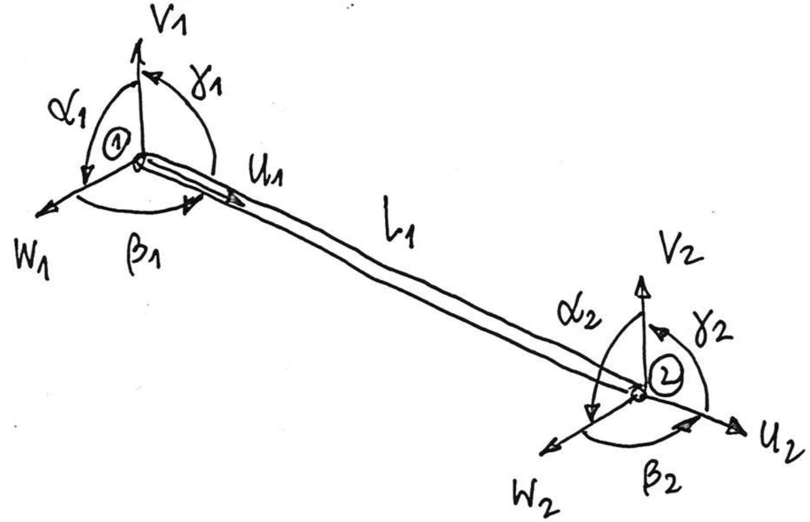
$$\underset{1\times 12}{\lfloor q \rfloor_1} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$

$$\underset{1\times 12}{\lfloor q_g \rfloor_1} = \lfloor u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2 \rfloor$$

$$q_1 = u_1, \quad q_2 = v_1, \quad q_3 = w_1, \quad q_4 = \gamma_1, \quad q_5 = -\beta_1, \quad q_6 = \alpha_1$$

$$q_7 = u_2, \quad q_8 = v_2, \quad q_9 = w_2, \quad q_{10} = \gamma_2, \quad q_{11} = -\beta_2, \quad q_{12} = \alpha_2$$



$$\underset{12\times 1}{\{q\}_1} = \underset{12\times 12}{[T_f]_1} \cdot \underset{12\times 1}{\{q_g\}_1}$$

$$\underset{12\times 12}{[T_f]_1} = \left[\begin{array}{cccccc|cccccc}
1 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 1 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & & & \underset{6\times 6}{[0]} & & & \\
0 & 0 & 0 & 0 & 0 & 1 & & & & & & \\
0 & 0 & 0 & 0 & -1 & 0 & & & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & & & & & & \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & \underset{6\times 6}{[0]} & & & & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 1 \\
 & & & & & & 0 & 0 & 0 & 0 & -1 & 0 \\
 & & & & & & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right] = [T_f]^T$$

$$\underset{12\times12}{[k_g]_1} = \underset{12\times12}{[T_f]_1^T} \cdot \underset{12\times12}{[k]_1} \cdot \underset{12\times12}{[T_f]_1}$$

$$a_1 = \frac{EA_1}{l_1}, \quad b_1 = \frac{12\,EJ_{31}}{l_1^3}, \quad c_1 = b_1, \quad d_1 = \frac{6\,EJ_{31}}{l_1^2}, \quad e_1 = d_1,$$

$$r_1 = \frac{2\,EJ_{31}}{l_1}, \quad s_1 = r_1, \quad t_1 = \frac{G\,J_{o1}}{l_1}, \quad A_1 = \frac{\pi\left(d_o^2 - (d_o - 2t)^2\right)}{4}$$

$$J_{31} = J_{\eta 1} = \frac{\pi}{64}\left(d_o^4 - (d_o - 2t)^4\right), \quad J_{o1} = 2 \cdot J_{31}$$

$$\underset{18\times18}{[k_g]_1^*} = \begin{bmatrix} [k_g]_1 & \underset{12\times6}{[0]} \\ \underset{6\times12}{[0]} & \underset{6\times6}{[0]} \end{bmatrix}$$

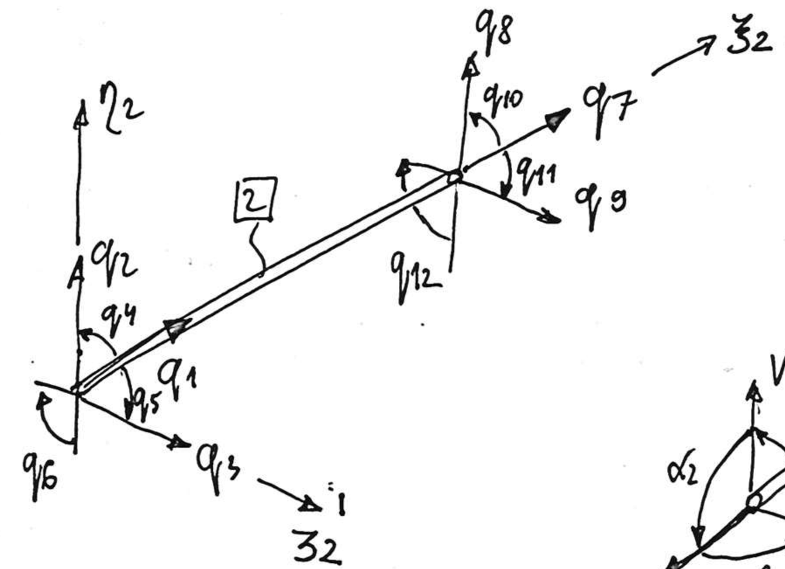
ELEMENT $\boxed{2}$

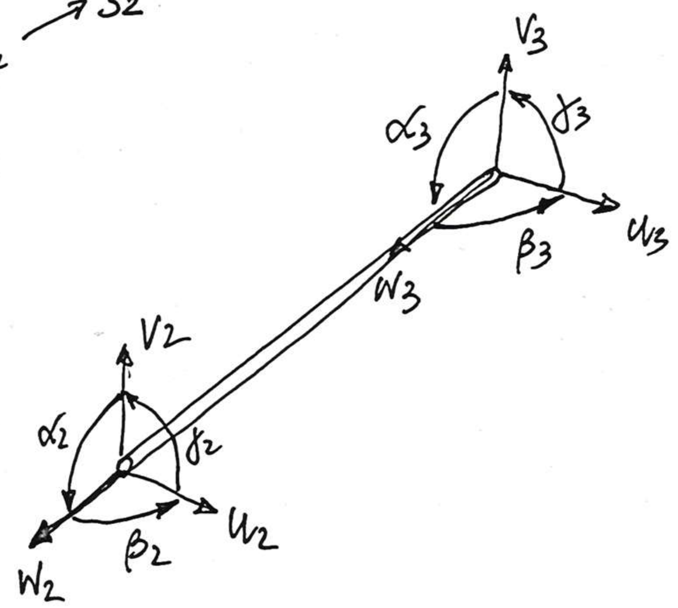
$$\lfloor q \rfloor_2 = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$

$$\lfloor q_g \rfloor_2 = \lfloor u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2, u_3, v_3, w_3, \alpha_3, \beta_3, \gamma_3 \rfloor$$

$$q_1 = -w_2, \quad q_2 = v_2, \quad q_3 = u_2, \quad q_4 = \alpha_2, \quad q_5 = -\beta_2, \quad q_6 = -\gamma_2$$

$$q_7 = -w_3, \quad q_8 = v_3, \quad q_9 = u_3, \quad q_{10} = \alpha_3, \quad q_{11} = -\beta_3, \quad q_{12} = -\gamma_3$$



$$\underset{12\times1}{\{q\}_2} = \underset{12\times12}{[T_f]_2} \cdot \underset{12\times1}{\{q_g\}_2}$$

$$[T_f]_2 = \left[\begin{array}{cccccc|cccccc}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & & & 0 & 0 & -1 & 0 & 0 & 0 \\
 & & & & & & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & \multicolumn{2}{c}{\underset{6\times6}{[0]}} & & & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & -1 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]$$

$$\underset{12\times12}{[k_g]_2} = \underset{12\times12}{[T_f]_2^T} \cdot \underset{12\times12}{[k]_2} \cdot \underset{12\times12}{[T_f]_2}$$

$$a_2 = \frac{EA_2}{l_2}, \quad b_2 = \frac{12EJ_{32}}{l_2^3}, \quad c_2 = \frac{12EJ_{\eta 2}}{l_2^3}, \quad d_2 = \frac{6EJ_{32}}{l_2^2}$$

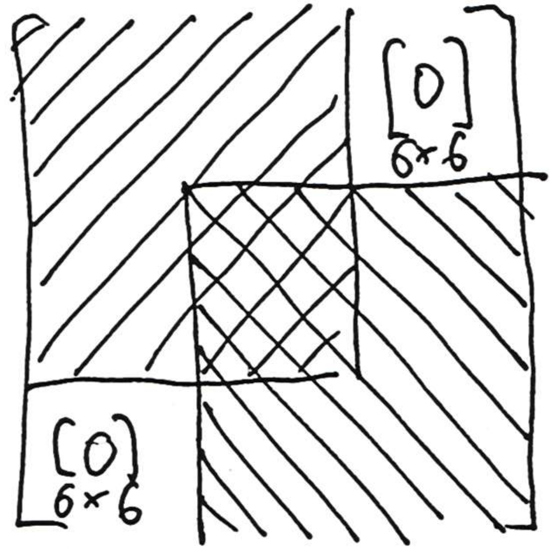
$$e_2 = \frac{6EJ_{\eta 2}}{l_2^2}, \quad r_2 = \frac{2EJ_{32}}{l_2}, \quad s_2 = \frac{2EJ_{\eta 2}}{l_2}, \quad t_2 = \frac{G\cdot J_{02}}{l_2}$$

$$J_{32} = \frac{hb^3}{12}, \quad J_{\eta 2} = \frac{bh^3}{12}, \quad J_{02} = 0.457\, bh^3$$

$$A_2 = b\cdot h$$

$$\underset{18\times18}{[k_g]_2^*} = \left[\begin{array}{c|c} \underset{6\times6}{[0]} & \underset{6\times12}{[0]} \\ \hline \underset{12\times6}{[0]} & [k_g]_2 \end{array}\right]$$

$$\underset{18\times 18}{[K]} = \underset{18\times 18}{[Kg]_1^*} + \underset{18\times 18}{[Kg]_2^*} =$$

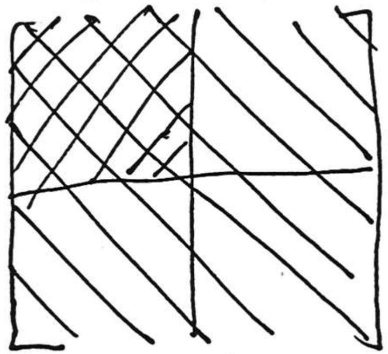


$$\underset{18\times18}{[K]}\cdot\underset{18\times1}{\{q\}}=\underset{18\times1}{\{F\}}$$

$u_1=0,\ v_1=0,\ w_1=0$

$\alpha_1=0,\ \beta_1=0,\ \gamma_1=0$

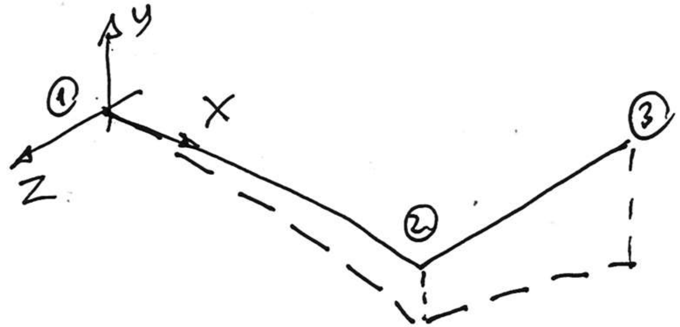
$$\underset{12\times12}{[K]}\cdot\underset{12\times1}{\{q\}}=\underset{12\times1}{\{F\}} \Rightarrow \underset{12\times1}{\{q\}}=\underset{12\times12}{[K]^{-1}}\cdot\underset{12\times1}{\{F\}}$$

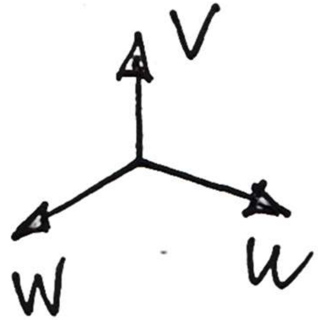


$$\underset{18\times18}{[K]}\cdot\underset{18\times1}{\{q\}}=\underset{18\times1}{\{F\}} \Rightarrow \text{REACTIONS}$$

DOF SOLUTION

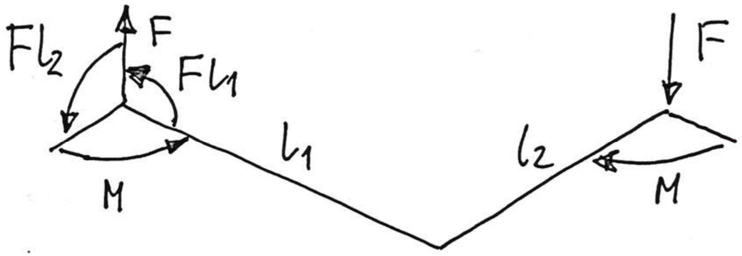
$$\{q\}_{12\times 1} = \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ \gamma_2 \\ u_3 \\ v_3 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} 0\ \text{mm} \\ -11.94\ \text{mm} \\ 14.93\ \text{mm} \\ -0.0243\ \text{rad} \\ -0.0249\ \text{rad} \\ -0.015\ \text{rad} \\ 29.07\ \text{mm} \\ -31.43\ \text{mm} \\ 14.93\ \text{mm} \\ -0.0269\ \text{rad} \\ -0.0527\ \text{rad} \\ -0.015\ \text{rad} \end{Bmatrix}$$



# REACTIONS

$$\underset{18\times1}{\{F\}} = \underset{18\times18}{[K]} \cdot \underset{18\times1}{\{q\}} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ M_{RX} \\ M_{RY} \\ M_{RZ} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -F \\ 0 \\ 0 \\ -M \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1000\,N \ (+F) \\ 0 \\ 750000 \ (F\cdot l_2) \\ 1000000 \ (M) \\ 1200000 \ (Fl_1) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -F \\ 0 \\ 0 \\ -M \\ 0 \end{Bmatrix}$$

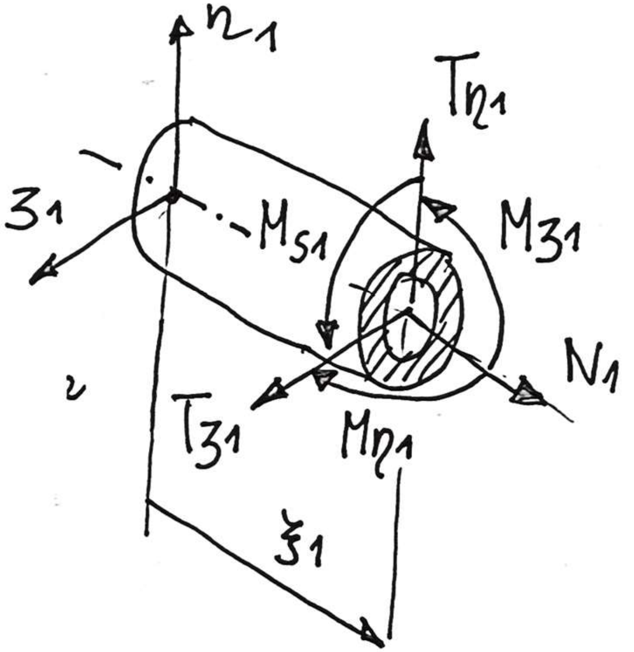



THE STRUCTURE IS IN EQUILIBRIUM

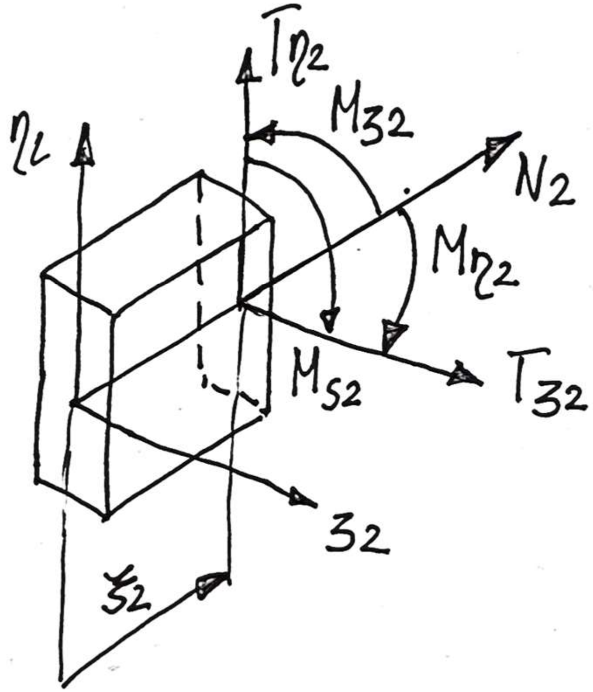
# ELEMENT SOLUTION

$$\underset{12\times 1}{\{q\}_i} = \underset{12\times 12}{[T_f]_i} \cdot \underset{12\times 1}{\{q_g\}_i} \quad , \quad i = 1, 2$$

$\boxed{1}$




$\boxed{2}$



AXIAL BAR: $(q_1, q_7)_i$

$$\varepsilon_{\xi i} = \frac{(q_7 - q_1)_i}{l_i}, \quad \sigma_{\xi i} = E \cdot \varepsilon_{\xi i}, \quad N_i = \sigma_{\xi i} A_i$$

BEAM:

I) BENDING IN $(\xi\eta)_i$ PLANE: $(q_2, q_4, q_8, q_{10})_i$

$$v_i(\xi_i) = \underset{1\times 4}{\lfloor N(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_4 \\ q_8 \\ q_{10} \end{Bmatrix}_i$$

$$M_{zi}(\xi_i) = E \cdot J_{zi} \cdot v_i''(\xi_i) =$$

$$= E \cdot J_{zi} \cdot \underset{1\times 4}{\lfloor N''(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_4 \\ q_8 \\ q_{10} \end{Bmatrix}_i$$

$$\sigma_{\xi i}^{I} = -\frac{M_{zi}(\xi_i)\cdot\eta_i}{J_{zi}}$$

$$T_{\eta i}(\xi_i) = -EJ_{zi}\cdot v_i'''(\xi_i) =$$

$$= -EJ_{zi}\cdot\underset{1\times4}{\left[N'''\right]}\cdot\begin{Bmatrix} q_2 \\ q_4 \\ q_8 \\ q_{10} \end{Bmatrix}_i = \text{const}$$

SHEAR STRESS CAUSED BY $T_{\eta i}(\xi_i)$ IS NEGLECTED.

$\underline{\mathrm{II}}$) BENDING IN $(\xi\zeta)$ PLANE : $(q_3, q_5, q_9, q_{11})_i$

$$w_i(\xi_i) = \underset{1\times 4}{\lfloor N(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_3 \\ q_5 \\ q_9 \\ q_{11} \end{Bmatrix}_i$$

$$M_{\eta i}(\xi_i) = E \cdot J_{\eta i}\, w_i''(\xi_i) =$$

$$= E J_{\eta i} \cdot \underset{1\times 4}{\lfloor N''(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_3 \\ q_5 \\ q_9 \\ q_{11} \end{Bmatrix}_i$$

$$\sigma_{\xi i}^{II} = - \frac{M_{\eta i}(\xi_i) \cdot z_i}{J_{\eta i}}$$

$$T_{zi}(\xi_i) = - E J_{\eta i} \cdot w_i'''(\xi_i) =$$

$$= - E J_{\eta i} \underset{1\times 4}{\lfloor N''' \rfloor} \cdot \begin{Bmatrix} q_3 \\ q_5 \\ q_9 \\ q_{11} \end{Bmatrix}_i = \text{const}$$

SHEAR STRESS CAUSED BY $T_{zi}(\xi_i)$ IS NEGLECTED.

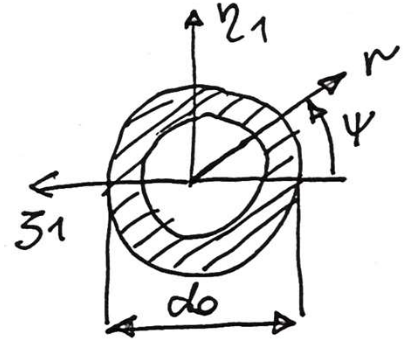
TORSION BAR : $(q_6, q_{12})_i$

$$\varphi_i(\xi_i) = \underset{1\times 2}{\lfloor N(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_i = \left\lfloor 1 - \frac{\xi_i}{l_i} \, , \; \frac{\xi_i}{l_i} \right\rfloor \cdot \begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_i =$$

$$= (q_6)_i + \frac{(q_{12} - q_6)_i}{l_i} \cdot \xi_i$$

$$\begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

ELEMENT $\boxed{1}$ :




$$\tau_1(r) = G\cdot\gamma_1(r) = \frac{E}{2(1+\nu)}\cdot\frac{d\varphi_1(\xi_1)}{d\xi_1}\cdot r =$$

$$= \frac{E}{2(1+\nu)}\cdot\left\lfloor -\frac{1}{L_1}\,,\ \frac{1}{L_1}\right\rfloor\cdot\begin{Bmatrix} q_6 \\ q_{12}\end{Bmatrix}_1\cdot r =$$

$$= \frac{E\,(q_{12}-q_6)_1}{2(1+\nu)\,L_1}\cdot r \qquad {}_{12}\ ,(q_6)_1 = 0$$

$$\tau_{1max} = \tau_1\left(\frac{d_0}{2}\right) = \frac{E\cdot(q_{12})_1 d_0}{4(1+\nu)\,L_1} = \frac{E\,d_0\,\alpha_2}{4(1+\nu)L_1}$$

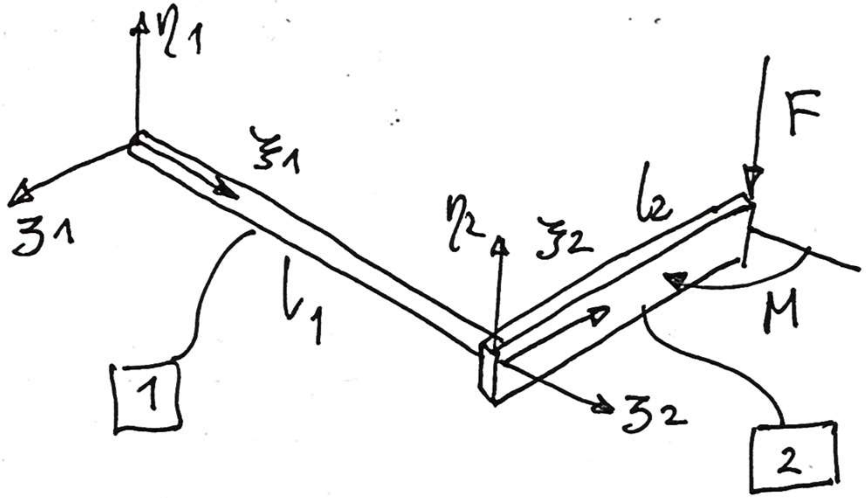
$$M_{S1} = \frac{\tau_1(r)\cdot J_{01}}{r} = \frac{E\cdot(q_{12})_1\cdot J_{01}}{2(1+\nu)\,L_1} = \frac{E\alpha_2\,J_{01}}{2(1+\nu)L_1} = \text{cost}$$

ELEMENT $\boxed{2}$ :

$$\begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_2 = \begin{Bmatrix} \gamma_2 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} -0.015 \text{ rad} \\ -0.015 \text{ rad} \end{Bmatrix} \Rightarrow \varphi_2(\xi_2) = q_6 = \text{const}$$

$$\Rightarrow \frac{d\,\varphi_2(\xi_2)}{d\xi_2} = 0 \quad \Rightarrow \quad \begin{array}{l} \tau_2 = 0 \\ M_{s2} = 0 \end{array}$$

## ELEMENT RESULTS




NORMAL FORCES:

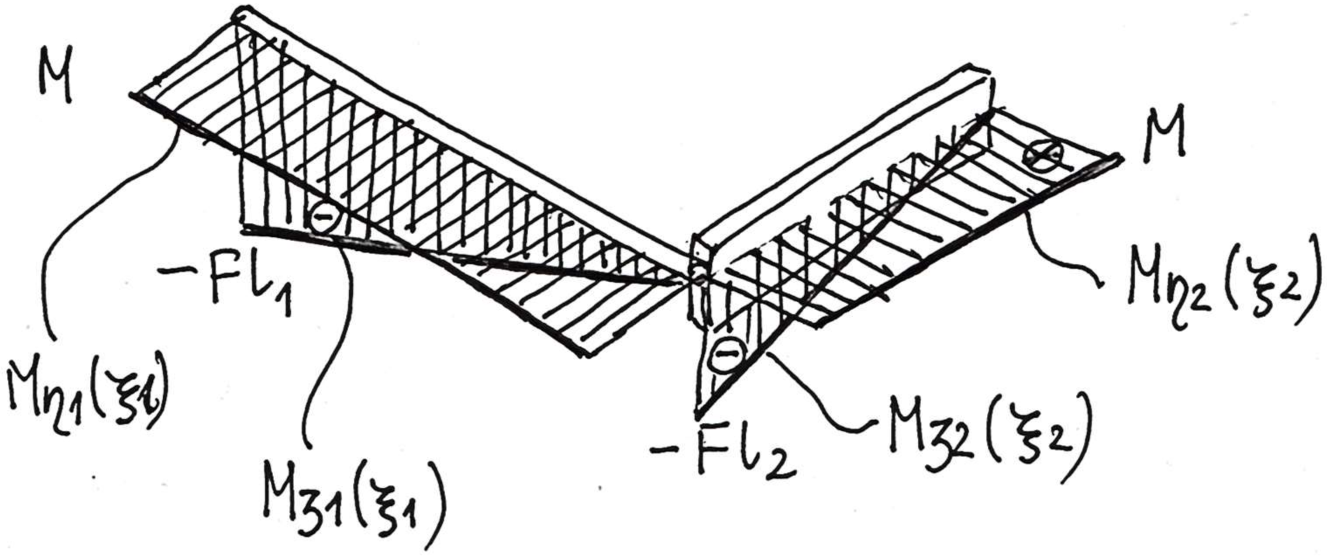
$$N_1 = 0 \quad , \quad N_2 = 0$$

SHEAR FORCES:

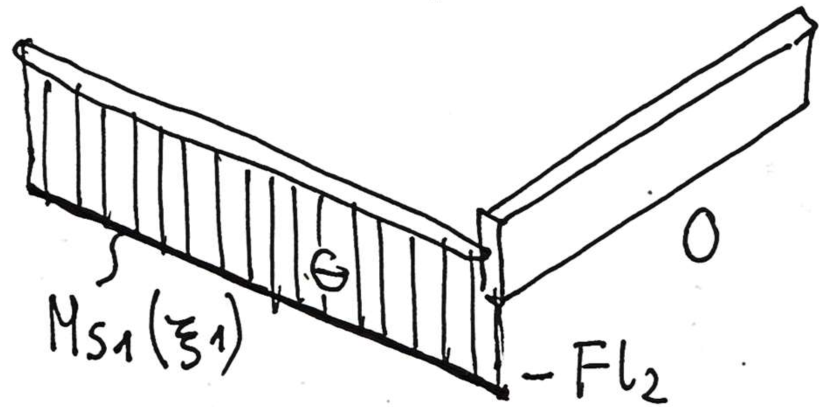
$$T_{\eta_1} = -F \quad , \quad T_{\eta_2} = -F$$

$$T_{\zeta_1} = 0 \quad , \quad T_{\zeta_2} = 0$$

BENDING MOMENTS :




TORQUE :



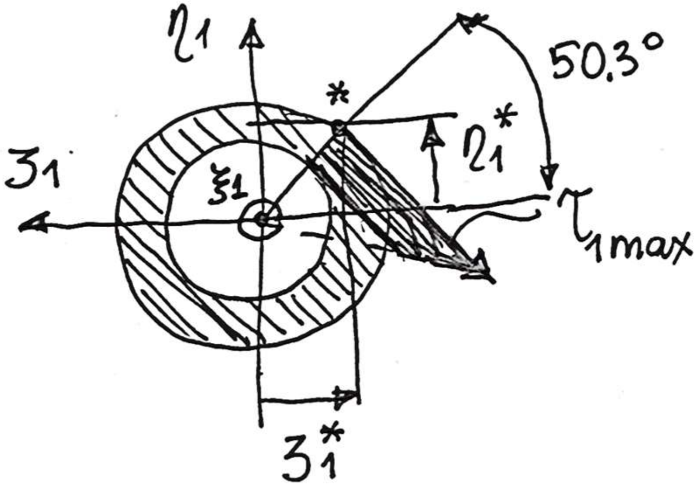
POINT OF THE HIGHEST STRESS:

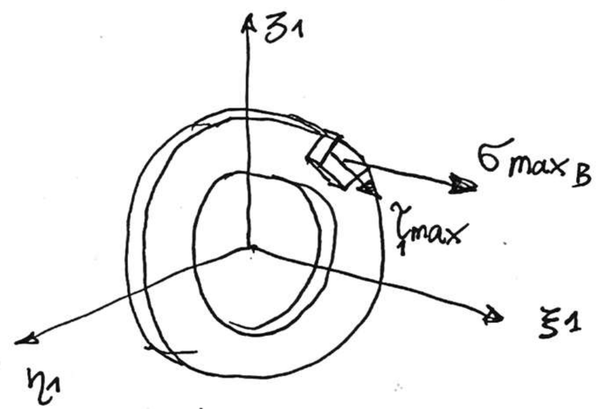
ELEMENT $\boxed{1}$ , $\xi_1 = 0$

NORMAL STRESS DUE TO BENDING:

$$\sigma_{MAX\,B} = \sigma_{\xi_1}^{I} + \sigma_{\xi_1}^{II} =$$

$$= -\frac{M_{\zeta 1}(0)\cdot \eta_1^*}{J_{\zeta 1}} - \frac{M_{\eta 1}(0)\cdot \zeta_1^*}{J_{\eta 1}} = 161.9\,\mathrm{MPa}$$




SHEAR STRESS

$$\tau_{1max} = \frac{E d_0 \alpha_2}{4(1+\nu)L_1} = -38.86 \text{ MPa}$$

MAXIMUM EQUIVALENT STRESS :

$$\sigma_{EQV} = \sqrt{\sigma_{MAX\,B}^2 + 3\,\tau_{1max}^2} = 175.34 \text{ MPa}$$

# Von Mises stress (SEQV) from ANSYS